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POLYTOPES OF CONSTANT WEIGHT

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ABSTRACT

A study is made of the non-regular planar 3-connected graphs with constant weight.

1. Introduction

An edge of a polytope is said to have weight n provided the sum of the valences of its vertices is n. Kotzig [4] proved that any 3-dimensional polytope (hereafter to be called 3-polytope) has an edge of weight at most 13. In this paper, we shall examine 3-polytopes of constant weight, that is, polytopes with all edges having the same weight. Using a theorem of Steinitz [7] which states that a graph is the graph of a 3-polytope if and only if it is planar and 3-connected, we can prove the existence of 3-polytopes of constant weight by finding planar 3-connected graphs of constant weight. The regular 3-connected planar graphs constitute a family of 3-polytopes with constant weight. Existence and facial structure of such 3-polytopes were studied by Hawkins et al. [2], Owens [6], Malkevitch [5] and others. In this paper we shall study the non-regular planar 3-connected graphs with constant weight.

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2. The main results

We shall call a 3-polytope an (a,b)-polytope if, for each edge, one of its vertices is a-valent and the other is b-valent with $a \leqq b$. If $a \neq b$, then the graph of an (a,b)-polytope must be bipartite and the polytope cannot have triangular

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faces. If p_i denotes the number of i-sided faces of the polytope, Euler's equation (see [1], pp. 236–7) yields:

$$\sum_{i\geq 3}(4-i)v_i+\sum_{i\geq 4}(4-i)p_i=8.$$

Thus, if $a\neq b$, the (a,b)-polytope must have 3-valent vertices. Let P be a $(3,b)$-polytope with v, e and f vertices, edges and faces respectively. It follows that $v=v_3+v_b$ and $e=3v_3=bv_b$, thus

$$e=\frac{3bv}{b+3}\quad\text{and}\quad f=2-v+e=2-v+\frac{3bv}{b+3}.$$

Now we have

$$\sum_{i\geq 4} ip_i=2e=\frac{6bv}{b+3}$$

and

$$\sum_{i\geq 4}4p_i=4f=8-4v+\frac{12bv}{b+3},$$

thus

$$\sum_{i\geq 4}(i-4)p_i=4v-8-\frac{6bv}{b+3}. \tag{1}$$

When $b\geqq 6$, the right hand of the last equation is negative, while the left hand is always non-negative, thus (a,b)-polytopes with $a\neq b$ exist only for $a=3$ and $b=4$ or 5.

(3, 4)-polytopes

If P is a $(3,4)$-polytope, then $3v_3=4v_4$ and $v=v_3+v_4=\frac{7v_4}{3}$, thus v is a multiple of 7. From (1) we see that no $(3,4)$-polytope can have 7 vertices. Fig. 1 shows the graph of a $(3,4)$-polytope with 14 vertices (the rhombic dodecahedron). To construct graphs of other $(3,4)$-polytopes, we take two adjacent

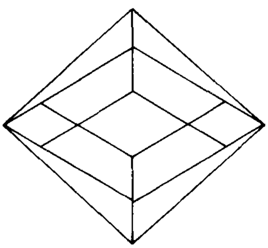

Fig. 1

4-sided faces in a given (3,4)-polytope and make the change indicated in Fig. 2. This change introduces 7 new vertices, retains planarity and 3-connectedness, thus the resulting 3-polytope is a (3,4)-polytope with 7 additional vertices having two adjacent 4-sided faces. This implies

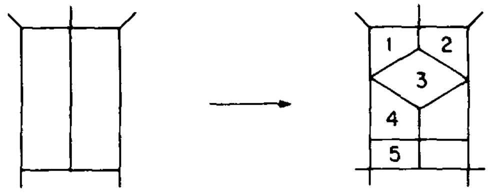

Fig. 2

THEOREM 1. *(3,4)-polytopes with 7m vertices exist for all* $m \geqq 2$.

To examine the face structure of (3,4)-polytopes we note that

$$e = \frac{1}{2}\sum ip_i, \quad v = \frac{7}{12}e = \frac{7}{24}\sum ip_i, \quad f = \sum p_i;$$

thus Euler's equation yields

$$\sum_{i \geq 4} (24 - 5i)p_i = 48. \tag{2}$$

Since the graph of a (3,4)-polytope is bipartite, $p_i = 0$ if i is odd. The following two theorems deal with realization by (3,4)-polytopes of sequences $\{p_{2k}\}$ that satisfy (2).

THEOREM 2. *There does not exist a (3,4)-polytope with* $p_{2k} = 1$ $(k \geqq 3, k$ *fixed*) *and* $p_4 = (24 + 10k)/4$.

PROOF. Assume that a (3,4)-polytope with $p_{2k} = 1$ and $p_4 = (24 + 10k)/4$ exists. In the planar representation of its graph, consider only the 4-valent vertices. We connect two 4-valent vertices by an edge if they belong to the same 4-sided face. The k 4-valent vertices on the boundary of the $2k$-sided face will be connected by edges to their immediate neighbors, forming a k-gon. The graph thus obtained will be planar, connected, all its countries but one will be triangular, all vertices, except the vertices on the boundary of the exceptional country, will be 4-valent, while the vertices on the boundary of the exceptional country will be 5-valent. By (2), k must be even, thus by removing $k/2$ disjoint edges from the boundary of the k-sided country, we would obtain a connected planar graph with the following properties:

(i) all vertices are 4-valent;

(ii) all countries but exactly one are triangular.

The nonexistence of such graphs was proved by Malkevitch [5] (Cor. 2.17). Thus the theorem is proved.

THEOREM 3. *If* $p_{2k} = 2r_k$ *and the sequence* $\{p_4, p_6, \cdots\}$ *satisfies* (2), *then there exist* (3, 4)*-polytopes realizing this sequence.*

PROOF. We start by choosing a chain of four 4-sided faces on the rhombic dodecahedron. By splitting the two internal faces as indicated in Fig. 2, the two end faces of the chain will become hexagons. By repeating the same type of face splitting of the two adjacent 4-gons (1,2 in Fig. 2), and continuing this process $k-2$ times, we obtain a (3, 4) polytope with $p_{2k} = 2$ and $p_4 = 5k$ ($k \geqq 2$). Since the same process can be applied to the chain of four 4-gons (2, 3, 4, 5 in Fig. 2), the theorem follows easily by induction on $\frac{1}{2}\Sigma p_{2k}$ ($k \geqq 3$).

(3, 5)*-polytopes*

By the above type of argument, any (3, 5)-polytope has $8m$ vertices and m cannot be less than 4. Fig. 3 gives the graphs of (3, 5)-polytopes with 32 vertices

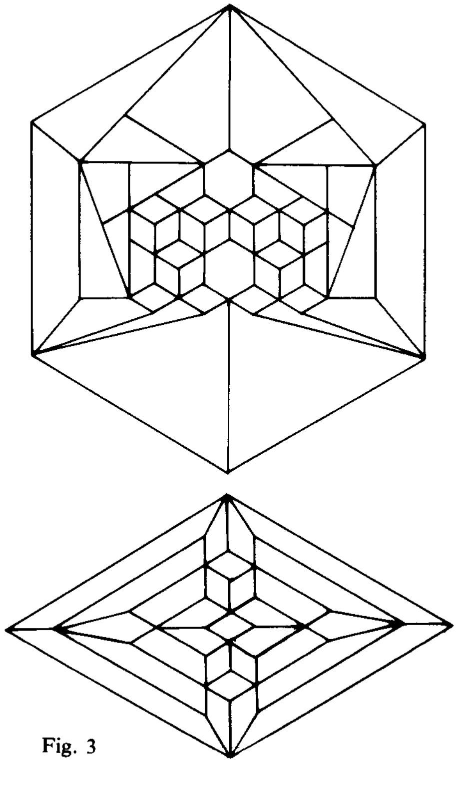

Fig. 3

(the rhombic triacontahedron) and 56 vertices. Fig. 4 shows how we can modify a planar 3-connected (3, 5)-graph with $8m$ to obtain one with $8(m+2)$ vertices, retaining the basis configuration of a chain of four 4-gons. This implies

THEOREM 4. *(3, 5)-polytopes with $8m$ vertices exist for $m = 4$ and $m \geqq 6$.*

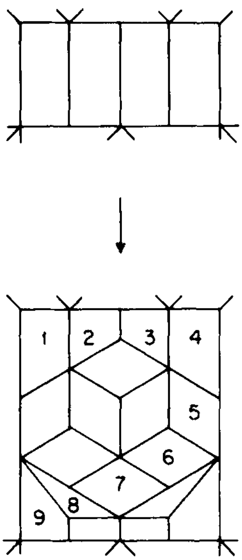

Fig. 4

To obtain the Eberhard type relation that faces of a (3, 5)-polytope satisfy note that

$$e = \frac{1}{2}\sum ip_i, \quad v = \frac{8}{15}e = \frac{8}{30}\sum ip_i, \quad \text{and} \quad f = \sum p_i.$$

Thus:

$$\sum_{i \geqq 4} (30 - 7i)p_i = 60.$$

THEOREM 5. *There does not exist a (3, 5)-polytope with $p_{2k} = 1$ ($k \geqq 3$ fixed) and $p_4 = 15 + 7k$.*

PROOF. Assume first that k is even. If such a (3, 5)-polytope exists, by considering only the 3-valent vertices of its planar graph representation the same type of construction as in Theorem 2 would imply the existence of a connected planar graph with the following properties:

(i) all countries but one are 5-sided, the exceptional country is k-sided.

(ii) all vertices are 3-valent, except the vertices of the k-gon which are 4-valent.

Since k is even, by removal of $k/2$ disjoint edges we would obtain a planar connected graph, regular of degree 3, all countries with exactly one exception are 5-sided. Such a graph does not exist. (Malkevitch [5], Lemma 2.8.)

If k is odd, by considering only the 5-valent vertices, and the above type construction, by removing $(k+1)/2$ edges from the exceptional country, one would obtain a planar connected graph with the following properties:

(i) all vertices but one are 5-valent;

(ii) all countries but one are triangular;

(iii) the exceptional vertex belongs to the exceptional country.

Such a graph does not exist (Malkevitch [5], Cor. 2.21). This concludes the proof of the theorem.

Corollary. *There does not exist a* $(3,5)$*-polytope with* 40 *vertices.*

Proof. The only possible sequence that satisfies (3) and

$$\sum p_i = f = 2 - v + e = 37 \text{ is } p_6 = 1, \quad p_4 = 36;$$

by Theorem 5 no $(3,5)$-polytope can realize this sequence.

Theorem 6. *If* $p_{2k} = 2r_k$, $k \geqq 2$, *and* $\{p_4, p_6, \cdots\}$ *satisfies* (3), *then this sequence is realizable by a* $(3,5)$*-polytope.*

Proof. The proof is similar to the proof of Theorem 3. We start by choosing a chain of six 4-gons in the rhombic triacontahedron. By splitting the four inner 4-gons as indicated in Fig. 4, and continued splitting of the chain of four 4-gons (1, 2, 3, 4 in Fig. 4), we obtain a $(3,5)$-polytope with $p_{2k} = 2$ $(k \geqq 3)$ and $p_4 = 14k$. The same process can now be applied to the chain of six 4-gons (4, 5, 6, 7, 8, 9 in Fig. 4), thus simple induction yields the proof of the theorem.

3. Remarks and problems

Some other sequence $\{p_i\}$ that satisfy (2) or (3) can be shown to be realizable by $(3,4)$- or $(3,5)$-polytopes, but a general necessary and sufficient condition is not known.

It is easy to show that the rhombic dodecahedron and the rhombic triacontahedron are the only $(3,4)$- and $(3,5)$-polytopes with 14 (32) vertices. We conjecture that all $(3,4)$-polytopes can be obtained from the rhombic dodecahedron by the face splitting described in Fig. 2. The question of generating all $(3,5)$-polytopes seems to be intractable.

Similar results could probably be obtained for graphs on orientable 2-manifolds of genus g; estimates for the weight of edges of such graphs are given by Jucovič [3].

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